RECENT RESULTS ON NEAR-BEST SPLINE QUASI-INTERPOLANTS

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Abstract

Roughly speaking, a near-best (abbr. NB) quasi-interpolant (abbr. QI) is an approximation operator of the form $Q_a f = \sum_{\alpha \in A} \Lambda_\alpha(f) B_\alpha$ where the B_α's are B-splines and the $\Lambda_\alpha(f)$'s are linear discrete or integral forms acting on the given function f. These forms depend on a finite number of coefficients which are the components of vectors a_α for $\alpha \in A$. The index a refers to this sequence of vectors. In order that $Q_a p = p$ for all polynomials p belonging to some subspace included in the space of splines generated by the B_α's, each vector a_α must lie in an affine subspace V_α, i.e. satisfy some linear constraints. However there remain some degrees of freedom which are used to minimize $\|a_\alpha\|_1$ for each $\alpha \in A$. It is easy to prove that $\max\{\|a_\alpha\|_1; \alpha \in A\}$ is an upper bound of $\|Q_a\|_\infty$: thus, instead of minimizing the infinite norm of Q_a, which is a difficult problem, we minimize an upper bound of this norm, which is much easier to do. Moreover, the latter problem has always at least one solution, which is associated with a NB QI. In the first part of the paper, we give a survey on NB univariate or bivariate spline QIs defined on uniform or non-uniform partitions and already studied by the author and coworkers. In the second part, we give some new results, mainly on univariate and bivariate integral QIs on *non-uniform* partitions: in that case, NB QIs are more difficult to characterize and the optimal properties strongly depend on the geometry of the partition. Therefore we have restricted our study to QIs having interesting shape properties and/or infinite norms uniformly bounded independently of the partition.

1. Introduction and notations

The spline quasi-interpolants (abbr. QIs) considered in this paper have the following general form:

$$Q_a f = \sum_{\alpha \in A} \Lambda_\alpha(f) B_\alpha.$$

A denotes a finite or infinite set of indices. B_α is a B-spline with support Σ_α, defined on a uniform or nonuniform partition. The index a of Q_a refers to a family of vectors $\{a_\alpha, \alpha \in A\}$ which is described below. The coefficients $\Lambda_\alpha(f)$ are discrete or integral functionals of the following types:

$$\Lambda_\alpha(f) = \sum_{\gamma \in \Gamma_\alpha} a_\alpha(\gamma) f(x_\gamma) \;\; or \;\; \Lambda_\alpha(f) = \sum_{\gamma \in \Gamma_\alpha} a_\alpha(\gamma) \int_{\Sigma_\gamma} B_\gamma^* f,$$

where Γ_α is a finite set of indices nearby α. In the discrete coefficient functionals $\Lambda_\alpha(f)$, the points x_γ lie in Σ_γ and the associated QIs are called *discrete quasi-interpolants* (abbr. dQIs). Those with integral coefficient functionals $\Lambda_\alpha(f)$ are called *integral quasi-interpolants* (abbr. iQIs): the weight function in the integral is a B-spline B_γ^* (which can be different from B_γ) satisfying $\int_{\Sigma_\gamma} B_\gamma^* = 1$ for all indices γ. For $\alpha \in A$, the vectors $a_\alpha = (a_\alpha(\gamma), \gamma \in \Gamma_\alpha)$ are determined in the following way:

(i) let $\mathbb{P}$ be a space of polynomials included in the space of splines generated by the family $\{B_\alpha, \alpha \in A\}$, then we impose that Q_a be exact on $\mathbb{P}$, i.e. $Q_a p = p$ for all $p \in \mathbb{P}$. This is equivalent to say that, for each $\alpha \in A$, the vector a_α belongs to some affine subspace V_α of $\mathbb{R}^{n_\alpha}$ where $n_\alpha = card(\Gamma_\alpha)$ is large enough.

(ii) in general there remain some undetermined components of a_α. Then we minimize the l^1-norm of this vector: using standard results in optimization, it is easy to prove that, for each $\alpha \in A$, there exists at least one vector $a_\alpha \in V_\alpha$ solution to this problem.

The infinite norm of Q_a is bounded above by $\nu(Q_a) = \max_{\alpha \in A} \nu(a_\alpha)$ where $\nu(a_\alpha) = \|a_\alpha\|_1$. Thus, instead of minimizing this norm, we minimize $\nu(a_\alpha)$ for all $\alpha \in A$, the vector a_α satisfying the constraints $a_\alpha \in V_\alpha$, which is much easier to do. When a_α^* is a solution of this minimization problem, we say that the corresponding QI $Q^* = Q_{a^*}$ is a *near-best quasi-interpolant* (abbr. NBQI). In the first part of the paper, we present various examples of NBQIs which have been studied in [1][22] and in the reports [2]-[4]. In the second part, we present new results on some families of univariate and bivariate iQIs: the proofs are somewhat sketched and will be detailed elsewhere [5][25][26].

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2. Univariate Near-best QIs on uniform partitions

Butzer et al. [10][11] have completed the results of Schoenberg [42] about the general expressions of spline dQIs of maximal approximation order (i.e. exact on polynomials having the same degree as the underlying spline) on the real line with $\mathbb{Z}$ as sequence of knots. Another technique, based on central difference operators, has been given by the author in [34].

2.1. **Near-best spline dQIs.** Let $M_{2m}(x)$ denote the centered cardinal B-spline. Consider the family of spline dQIs of order $2m$ depending on $n+1$ arbitrary parameters $a=(a_0,a_1,\ldots,a_n)$, $n\geq m$:

$$Q_a f=\sum_{i\in\mathbb{Z}}\Lambda f(i)M_{2m}(x-i),$$

with coefficient functionals

$$\Lambda f(i)=a_0 f(i)+\sum_{j=1}^{n}a_j\left(f(i+j)+f(i-j)\right).$$

Setting $\nu(a)=|a_0|+\sum_{j=1}^{n}|a_j|$, then we have $\|Q_a\|_\infty\leq\nu(a)$. By imposing that Q_a be exact on the space Π_r of polynomials of degree at most r, with $0\leq r\leq 2m-1$, we obtain a set of linear constraints: $a\in V_r\subset\mathbb{R}^{n+1}$. We say that $Q^*=Q_{a^*}$ is a *near best dQI* if

$$\nu(a^*)=\min\{\nu(a);a\in V_r\}.$$

There is existence, but in general not unicity, of solutions (see [2], [22]).
Example: cubic splines [22]. There is a unique optimal solution for $r=3$ and $n\geq 2$:

$$a_0^*=1+\frac{1}{3n^2},\quad a_n^*=-\frac{1}{6n^2},\quad a_j^*=0\ \ for\ \ 1\leq j\leq n-1.$$

Moreover, for all $n\geq 4$, $\|Q^*\|_\infty\leq 1+\frac{2}{3n^2}$. Here are the first values of $\|Q^*\|_\infty$ & $\nu(a^*)$; $n=1:1.222$ & 1.666; $n=2:1.139$ & 1.166; $n=3:1.074$ & 1.074.

2.2. **Near-best spline iQIs.** A similar study can be done for integral spline QIs. We refer to [2][22] and we only give an example given in these papers. Setting $a=(a_0,a_1,\ldots,a_n)$, $n\geq m$, $M_i(x)=M_{2m}(x-i)$ and $\langle f,M_i\rangle=\int fM_i$, we consider $Q_af=\sum_{i\in\mathbb{Z}}\Lambda f(i)M_i$ with coefficient functionals

$$\Lambda f(i)=a_0\langle f,M_i\rangle+\sum_{j=1}^{n}a_j\left(\langle f,M_{i-j}\rangle+\langle f,M_{i+j}\rangle\right).$$

As in section 4.2, we have $\|Q_a\|_\infty\leq\nu(a)$ and we say that $Q^*=Q_{a^*}$ is a *near best iQI* if $\nu(a^*)=\min\{\nu(a);a\in V_r\}$. There is existence, but in general not unicity, of solutions.

Example: cubic splines [22]. There is a unique optimal solution for $r = 3$ and $n \geq 2$:

$$a_0^* = 1 + \frac{2}{3n^2}, \quad a_n^* = -\frac{1}{3n^2}, \quad a_j^* = 0 \;\; for \;\; 1 \leq j \leq n-1.$$

Moreover, for all $n \geq 4$, $\|Q^*\|_\infty \leq 1 + \frac{4}{3n^2}$. Here are the first values of $\|Q^*\|_\infty$ & $\nu(a^*)$; $n = 1 : 1.5278$ & 2.333; $n = 2 : 1.2778$ & 1.333; $n = 3 : 1.1481$ & 1.1482.

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3. Bivariate Near-best QIs on uniform partitions

3.1. **A general construction of dQIs.** Let φ be some bivariate B-spline on one of the two classical three or four direction meshes of the plane (e.g. box-splines or H-splines, see [3][8][6][9][14][31][34] citeSab6[41]). Let $\Sigma = supp(\varphi)$ and $\Sigma^* = \Sigma \cap \mathbb{Z}^2$. Let a be the hexagonal (or lozenge=rhombus) sequence formed by the values $\{\varphi(i), i \in \Sigma^*\}$. The associated central difference operator $\mathcal{D}$ is an isomorphism of $\mathbb{P}(\varphi)$, the maximal subspace of "complete " polynomials in the space of splines $\mathcal{S}(\varphi)$ generated by the integer translates of the B-spline φ (see e.g. [6][9]). Computing the expansion of a in some basis of the space of hexagonal (or lozenge) sequences amounts to expand $\mathcal{D}$ in some basis of central difference operators. Then, computing the formal inverse $\mathcal{D}^{-1}$ allows to define the dQI

$$\mathcal{Q}f = \sum_{k \in \mathbb{Z}^2} \mathcal{D}^{-1} f(k) \varphi(\cdot - k)$$

which is exact on $\mathbb{P}(\varphi)$. Let us now give two examples of NB dQIs which are detailed in [22]. The definition of these operators is quite the same as in section 2.1.

3.2. **Near-best spline dQIs on a three direction mesh.** For example, let φ be the C^2 quartic box-spline with support $\Sigma = H_2$ where H_s denotes the regular hexagon with edges of length $s \geq 1$, centered at the origin and $H_s^* = H_s \cap \mathbb{Z}^2$. The near-best dQIs, which are exact on Π_3, have coefficient functionals with supports consisting of the center and of the 6 vertices of $H_s^*, s \geq 1$. The coefficients of values of f at those points are respectively $1 + \frac{1}{2s^2}$ and $-\frac{1}{12s^2}$, therefore the infinite norm of the optimal dQIs Q_s^* is bounded above by $\nu_s^* = 1 + \frac{1}{s^2}$. Here are the first values of $\|Q^*\|_\infty$ & ν_s^*; $n = 1 : 1.34028$ & 2; $n = 2 : 1.22917$ & 1.25; $n = 3 : 1.10185$ & 1.111.

3.3. **Near-best spline dQIs on a four direction mesh.** For example, let φ be the C^1 quadratic box-spline. Let Λ_s be the lozenge (rhombus) with edges of length $s \geq 1$, centered at the origin, and let $\Lambda_s^* = \Lambda_s \cap \mathbb{Z}^2$. The near-best dQIs which are exact on Π_2 have coefficient functionals with supports consisting of the center and the 4 vertices of $\Lambda_s^*, s \geq 1$. The coefficients of values of f at those points are respectively $1 + \frac{1}{2s^2}$ and

$-\frac{1}{8s^2}$, therefore the infinite norm of the optimal dQIs Q_s^* is bounded above by $\nu_s^* = 1+\frac{1}{s^2}$. Here are the first values of $\|Q^*\|_\infty$ & ν_s^*; $n = 1 : 1.5$ & 2; $n = 2 : 1.25$ & 1.25; $n = 3 : 1.111$ & 1.111.

Examples with C^2 quartic box-splines are also given in [22].

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4. Univariate Near-best QIs on non-uniform partitions

In this section, we consider dQIs or iQIs of degree $m \geq 2$ defined on a bounded interval $I = [a, b]$. Let $E_m = \{-m+1, \dots, 0\}$ and $J = \{0, 1, \dots, m+n-1\}$. Let $T = \{t_i; i \in E_m \cup J\}$ be an arbitrary non-uniform increasing sequence of knots with multiple knots at $t_0 = a$ and $t_n = b$, as usual. Let B_j be the B-spline with support $[t_{j-m}, t_{j+1}]$ for $j \in J$, and let $e_p(x) = x^p$ for all $p \geq 0$. Setting $\theta_j = \frac{1}{m}\sum_{s\in E_m} t_{j+s}$, it is well known that $e_1 = \sum_{j\in J}\theta_j B_j$ and $e_2 = \sum_{j\in J}\theta_j^{(2)} B_j$, with $\theta_j^{(2)} = \frac{2}{m(m-1)}\sum_{(r,s)\in E_m^2, r<s} t_{j+r}t_{j+s}$. We recall the following expansion [27]:

$$\lambda_j = \theta_j^2 - \theta_j^{(2)} = \frac{1}{m^2(m-1)} \sum_{(r,s)\in E_m^2, r<s} (t_{j+r} - t_{j+s})^2 > 0.$$

More generally $e_r = \sum_{j\in J}\theta_j^{(r)} B_j$ where the $\theta_j^{(r)}$'s are proportional to symmetric functions of knots. The simplest dQI is the Schoenberg operator

$$S_1 f = \sum_{i\in J} f(\theta_i) B_i$$

which is exact on Π_1 and is also shape preserving. Moreover, it satisfies

$$S_1 e_2 - e_2 = \sum_{i\in J} \lambda_j B_i \geq 0.$$

4.1. **Uniformly bounded dQIs.** Let us only give an example: we start from a family of differential QIs of degree m which are *exact on* Π_2 (see also [4]):

$$S_2^* f = \sum_{j\in J} \lambda_j^{(2)}(f) B_j, \quad \lambda_j^{(2)}(f) = f(\theta_j) - \frac{1}{2}\lambda_j D^2 f(\theta_j).$$

On the other hand, $\frac{1}{2}D^2 f(\theta_j)$ can be replaced on the space Π_2 by the second order divided difference $[\theta_{j-1}, \theta_j, \theta_{j+1}]f$, therefore the dQI defined by

$$S_2 f = \sum_{j\in J} \mu_j^{(2)}(f) B_j, \;\; \mu_j^{(2)}(f) = f(\theta_j) - \lambda_j[\theta_{j-1}, \theta_j, \theta_{j+1}]f$$

is also *exact on* Π_2. Moreover, one can write

$$\mu_i^{(2)}(f) = a_i f_{i-1} + b_i f_i + c_i f_{i+1}$$

with $a_i = -\lambda_i/\Delta\theta_{i-1}(\Delta\theta_{i-1}+\Delta\theta_i)$, $c_i = -\lambda_i/\Delta\theta_i(\Delta\theta_{i-1}+\Delta\theta_i)$, and $b_i = 1 + \lambda_i/\Delta\theta_{i-1}\Delta\theta_i$. So, according to the introduction

$$\|S_2\|_\infty \leq \max_{i\in J}(|a_i|+|b_i|+|c_i|) \leq 1 + 2\max_{i\in J}\frac{\lambda_i}{\Delta\theta_{i-1}\Delta\theta_i}.$$

The following theorem [4] extends a result given for quadratic splines in [22][33].

Theorem 1. *For any degree m, the dQIs S_2 are uniformly bounded independently of the partition. More specifically, if $[r]$ denotes the floor of r:*

$$\|S_2\|_\infty \leq [\frac{1}{2}(m+4)]$$

Remark. For quadratic splines, one can prove that $\|S_2\|_\infty \leq 2.5$ for all partitions. For uniform partitions, one gets $\|S_2\|_\infty = \frac{305}{207} \approx 1.4734$

4.2. **Near-best dQIs.** Let us consider the family of dQIs of degree m defined by

$$Qf = Q_{p,q}f = \sum_{i\in\mathbb{Z}} \mu_i(f)B_i.$$

depending on the two integer parameters $p \geq m$ and $q \leq \min(m, 2p)$. Their coefficient functionals depend on $2p+1$ parameters

$$\mu_i(f) = \sum_{s=-p}^{p} \lambda_i(s) f(\theta_{i+s}),$$

and we impose that Q is *exact on the space* Π_q. The latter condition is equivalent to $Qe_r = e_r$ for all monomials of degrees $0 \leq r \leq q$. It implies that for all indices i, the parameters $\lambda_i(s)$ satisfy the system of $q+1$ linear equations:

$$\sum_{s=-p}^{p} \lambda_i(s)\theta_{i+s}^r = \theta_i^{(r)}, \quad 0 \leq r \leq q.$$

The matrix $V_i \in \mathbb{R}^{(q+1)\times(2p+1)}$ of this system, with coefficients $V_i(r,s) = \theta_{i+s}^r$, is a Vandermonde matrix of maximal rank $q+1$, therefore there are $2p-q$ *free parameters*. Denoting by $b_i \in \mathbb{R}^{q+1}$ the vector in the right hand side, with components $b_i(r) = \theta_i^{(r)}$, $0 \leq r \leq q$, we consider the sequence of minimization problems, for $i \in \mathbb{Z}$:

$$\min\{\|\lambda_i\|_1;\ \ V_i\lambda_i = b_i\}.$$

We have seen in the introduction that $\nu_1^*(Q) = \max_{i\in\mathbb{Z}} \min \|\lambda_i\|_1$ is an upper bound of $\|Q_q\|_\infty$ which is easier to evaluate than the true norm of the dQI. The *objective function being convex* and *the domains being affine subspaces*, we have the following

Theorem 2. *The above minimization problems have always solutions, which, in general, are non unique.*

Let us give an example of optimal dQIs ([1][4][22]) in order to bring out the constraints on the partition implied by the optimal property. For $m = 2$ and $p \geq 2$, assume that the partition T satisfies, for all i

$$\theta_{i-1} + \theta_i \leq \theta_{i-p} + \theta_{i+p} \leq \theta_i + \theta_{i+1},$$

then there is a unique optimal solution (here $h_i = t_i - t_{i-1}$):

$$\lambda_i^*(-p) = -\frac{1}{4}\frac{h_i^2}{(\theta_{i-p} - \theta_{i+p})(\theta_i - \theta_{i+p})}, \quad \lambda_i^*(p) = -\frac{1}{4}\frac{h_i^2}{(\theta_{i-p} - \theta_{i+p})(\theta_{i+p} - \theta_i)}$$

$$\lambda_i^*(p) = 1 + \frac{1}{4}\frac{h_i^2}{(\theta_{i+p} - \theta_i)(\theta_i - \theta_{i-p})}, \quad \lambda_i^*(s) = 0 \;\; for \;\; s \neq 0, -p, p.$$

Denote by $Q_{p,2}^*$ the associated dQI, then we have [4]

Theorem 3. *The infinite norm of $Q_{p,2}^*$ is uniformly bounded independently of p and of the partition T:*

$$\|Q_{p,2}^*\|_\infty \leq 3.$$

4.3. **Uniformly bounded iQIs.** Various types of integral QIs are considered in [15][40][5]. Here, we restrict our study to Goodman-Sharma (abbr. GS) type iQIs which appear in [21]. They are simpler than those we have already studied in [40]. Given $I = [a, b]$ and the sequence of knots $T = \{t_{-m} = \ldots = t_0 = a < t_1 < \ldots < t_{n-1} < b = t_n = \ldots = t_{n+m}\}$, let B_i be the B-spline of degree m and support $[t_{i-m}, t_{i+1}]$ normalized by $\sum_{i=0}^{n+m-1} B_i = 1$, and let $\tilde{M}_{i-1}(t)$ be the B-spline of degree $m-2$ with support $\tilde{\Sigma}_{i-1} = [t_{i-m+1}, t_i]$, normalized by $\tilde{\mu}_i^{(0)} = \tilde{\mu}_i(e_0) = \int_a^b \tilde{M}_{i-1}(t) = 1$. The original GS-type iQI can be written as follows

$$G_1 f = \sum_{i=0}^{n+m-1} \tilde{\mu}_i(f) B_i,$$

where $\tilde{\mu}_0(f) = f(t_0)$, $\tilde{\mu}_{n+m-1}(f) = f(t_n)$ and, for $1 \leq i \leq n+m-2$,

$$\tilde{\mu}_i(f) = \int_a^b \tilde{M}_{i-1}(t) f(t) dt.$$

It is easy to verify that G_1 is *exact on* Π_1 and that $\|G_1\|_\infty = 1$. It has also interesting *shape preserving properties*: for example it is proved in [21] that G_1 preserves the *positivity and* *the convexity* of the approximated function f. Moreover, the authors also prove that

$$G_1 e_2 - e_2 = \frac{2m}{m+1}(S_1 e_2 - e_2).$$

Let us also prove that it preserves the *monotonicity* of f.

Theorem 4. *If f is a monotone function on I, then $G_1 f$ is also monotone with the same sense of variation.*

Proof. Let us use the simplified notation $G_1 f = \sum_{i=0}^{n+m-1} \tilde{\mu}_i B_i$. Then the first derivative is $DG_1 f = m \sum_{i=1}^{n+m-1} \frac{\tilde{\mu}_i - \tilde{\mu}_{i-1}}{t_i - t_{i-m}} B_i^*$, where B_i^* is the B-spline of degree $m-1$, normalized as B_i. Assuming that f is monotone increasing, we have to prove that $DG_1 f \geq 0$. For that, it is sufficient to prove that the sequence $\{(\tilde{\mu}_i - \tilde{\mu}_{i-1}), 1 \leq i \leq n+m-1\}$ is monotone increasing. For $i = 1$, we have $\tilde{\mu}_1 - \tilde{\mu}_0 = \int_{t_0}^{t_1} \tilde{M}_0 f - f(t_0)$; as $\int_{t_0}^{t_1} \tilde{M}_0 = 1$, the mean value theorem gives $\int_{t_0}^{t_1} \tilde{M}_0 f = f(\xi_0)$ for some $\xi_0 \in [t_0, t_1]$, hence $\tilde{\mu}_1 - \tilde{\mu}_0 = f(\xi_0) - f(t_0) \geq 0$. The same kind of proof holds for $i = n+m-1$. Now, for $2 \leq i \leq n+m-2$, we know that $\tilde{M}_i - \tilde{M}_{i-1} = -DB_{i-2}^* =$ (see e.g. [7]), therefore $\tilde{\mu}_i - \tilde{\mu}_{i-1} = \frac{1}{t_i - t_{i-m}} \int (-DB_{i-2}^* f)$ and, after integration by parts, $\tilde{\mu}_i - \tilde{\mu}_{i-1} = \frac{1}{t_i - t_{i-m}} \int B_{i-2}^* f' \geq 0$.

Therefore, we can see that the operator G_1 is very close to the Schoenberg's operator. ¿From the expression of $G_1 e_2 - e_2$, we can deduce the family of GS-type iQIs defined by

$$G_2 f = f(t_0) B_0 + \sum_{i=1}^{n+m-2} [a_i \tilde{\mu}_{i-1}(f) + b_i \tilde{\mu}_i(f) + c_i \tilde{\mu}_{i+1}(f)] B_i + f(t_n) B_{n+m-1},$$

which are *exact* on Π_2. The three constraints $G_2 e_k = e_k, \quad k = 0, 1, 2$, lead to the following system of equations, for $1 \leq i \leq n+m-2$:

$$a_i + b_i + c_i = 1, \quad \theta_{i-1} a_i + \theta_i b_i + \theta_{i+1} c_i = \theta_i, \quad \tilde{\mu}_{i-1}^{(2)} a_i + \tilde{\mu}_i^{(2)} b_i + \tilde{\mu}_{i+1}^{(2)} c_i = \theta_i^{(2)}.$$

This is a consequence of the following facts

$$\tilde{\mu}_i(e_1) = \int_a^b t \tilde{M}_{i-1}(t) dt = \frac{1}{m} \sum_{s=1}^{m} t_{i-m+s} = \theta_i,$$

$$\begin{aligned} \tilde{\mu}_i^{(2)} = \mu_i(e_2) &= \int_a^b t^2 \tilde{M}_{i-1}(t) dt = \frac{2}{m(m+1)} \tilde{s}_2(T_i) \\ &= \frac{2}{m(m+1)} \sum_{1 \leq r \leq s \leq m} t_{i-m+r} t_{i-m+s} \end{aligned}$$

For $m = 2$ (quadratic splines), here are the explicit expressions of the coefficients:

$$a_i = \frac{-h_i^2}{(h_{i-1} + h_i)(h_{i-1} + h_i + h_{i+1})}, \quad c_i = \frac{-h_i^2}{(h_{i-1} + h_i + h_{i+1})(h_i + h_{i+1})},$$

and $b_i = 1 - a_i - c_i$. It is clear that $\|G_2\|_\infty \leq 1 + 2\max(|a_i| + |c_i|) \leq 5$. In fact, this upper bound is valid for any degree m: the computation of coefficients and the proof of this interesting result will be given in [5].

Theorem 5. *The iQIs G_2 are uniformly bounded independently of the partition and of the degree m. More specifically, one has*

$$\|G_2\|_\infty \leq 5$$

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5. Bivariate quadratic spline QIs on non-uniform partitions

At the author's knowledge, the only bivariate box-splines which have been extended to non uniform partitions of the plane are C^1-quadratic box-splines on criss-cross triangulations [13][30]. Recently [36]-[39] we have constructed a set of B-splines generating the space of quadratic splines on a rectangular domain and *having their support in this domain*. Moreover, we have defined a discrete quasi-interpolant which is exact on Π_2 and uniformly bounded independently of the partition (it is different from the operator introduced in [13], see e.g. [16],[17]). For the sake of simplicity, we assume here that the domain is the whole plane endowed with a nonuniform criss-cross triangulation obtained by drawing diagonals in each rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ of the partition defined by the two sequences $X = \{x_i,\ i \in \mathbb{Z}\}$ and $Y = \{y_j,\ j \in \mathbb{Z}\}$. We set $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $s_i = \frac{1}{2}(x_{i-1} + x_i)$ and $t_j = \frac{1}{2}(y_{j-1} + y_j)$. Let B_{ij} be the B-spline whose octagonal support Σ_{ij} is centered at the point $\omega_{ij} = (s_i, t_j)$. Let Π_{ij} be the continuous piecewise affine pyramid (of egyptian type) satisfying $\Pi_{ij}(\omega_{ij}) = 1$, whose support is the central rectangle R_{ij} of Σ_{ij}. As $\int_{R_{ij}} \Pi_{ij} = \frac{1}{3} h_i k_j$, we can also define the pyramid $\tilde{\Pi}_{ij} = \frac{3}{h_i k_j} \Pi_{ij}$ normalized by $\int_{R_{ij}} \tilde{\Pi}_{ij} = 1$. Finally, in the same way, let χ_{ij} be the characteristic function of R_{ij} and let $\tilde{\chi}_{ij} = \frac{1}{h_i k_j} \chi_{ij}$ normalized by $\int_{R_{ij}} \tilde{\chi}_{ij} = 1$. For monomials, we use the notation $e_{rs}(x, y) = x^r y^s$. We know study some families of discrete and integral quasi-interpolants. Detailes proofs will be given elsewhere.

5.1. **Discrete quasi-interpolants.** The simplest dQI is the Schoenberg operator defined by

$$S_1 f = \sum_{(i,j) \in \mathbb{Z}^2} f(\omega_{ij}) B_{ij}.$$

It is well known ([13], [16]) that $S_1 e_{rs} = e_{rs}$ for $0 \le r, s \le 1$ and

$$S_1 e_{20} = e_{20} + \frac{1}{4} \sum_{(i,j)\in\mathbb{Z}^2} h_i^2 B_{ij}, \quad S_1 e_{02} = e_{02} + \frac{1}{4} \sum_{(i,j)\in\mathbb{Z}^2} k_j^2 B_{ij}.$$

S_1 is clearly a positive operator and it preserves the bimonotonicity and the biconvexity of f (i.e. the monotonicity and the convexity in the directions of coordinate axes). This can be proved by using the expressions of partial derivatives $\partial_1 B_{ij}$ and $\partial_2 B_{ij}$ which are piecewise affine functions whose values are given in the technical report [39].

5.2. **Integral quasi-interpolants.** We now study the two following iQIs:

$$T_1 f = \sum_{(i,j)\in\mathbb{Z}^2} \langle f, \tilde{\Pi}_{ij} \rangle B_{ij}$$

where $\langle f, g \rangle = \int_{\mathbb{R}^2} f(v) g(v) dv$, and the GS-type iQI:

$$G_1 f = \sum_{(i,j)\in\mathbb{Z}^2} \langle f, \tilde{\chi}_{ij} \rangle B_{ij}.$$

Let $\mu_{ij}(f) = \frac{1}{h_i k_j} \int_{R_{ij}} f$ be the mean value of f on R_{ij}, then one can also write

$$G_1 f = \sum_{(i,j)\in\mathbb{Z}^2} \mu_{ij}(f) B_{ij}.$$

These two operators are very close to each other and to the Schoenberg operator defined before. In particular, they are exact on bilinear polynomials and they satisfy respectively

$$T_1 e_{20} = e_{20} + \frac{3}{10} \sum_{(i,j)\in\mathbb{Z}^2} h_i^2 B_{ij}, \quad T_1 e_{02} = e_{02} + \frac{3}{10} \sum_{(i,j)\in\mathbb{Z}^2} k_j^2 B_{ij},$$

$$G_1 e_{20} = e_{20} + \frac{1}{3} \sum_{(i,j)\in\mathbb{Z}^2} h_i^2 B_{ij}, \quad G_1 e_{02} = e_{02} + \frac{1}{3} \sum_{(i,j)\in\mathbb{Z}^2} k_j^2 B_{ij}.$$

Moreover, as S_1, they both preserve the bimonotonicity and the biconvexity of f.

¿From the properties of T_1 and G_1 on monomials e_{20} and e_{02}, one can deduce the two following iQIs which are both exact on the space Π_2 of bivariate quadratic polynomials:

$$T_2 f = \sum_{(i,j)\in\mathbb{Z}^2} \langle f, M_{ij} \rangle B_{ij}, \quad G_2 f = \sum_{(i,j)\in\mathbb{Z}^2} \langle f, \psi_{ij} \rangle B_{ij}.$$

where the two functions M_{ij} and ψ_{ij} are respectively defined by

$$M_{ij} = a_i \tilde{\Pi}_{i-1,j} + \bar{a}_i \tilde{\Pi}_{i+1,j} + b_{ij} \tilde{\Pi}_{ij} + c_j \tilde{\Pi}_{i,j-1} + \bar{c}_j \tilde{\Pi}_{i,j+1}$$

$$\psi_{ij} = \alpha_i \tilde{\chi}_{i-1,j} + \bar{\alpha}_i \tilde{\chi}_{i+1,j} + \beta_{ij} \tilde{\chi}_{ij} + \gamma_j \tilde{\chi}_{i,j-1} + \bar{\gamma}_j \tilde{\chi}_{i,j+1}$$

The coefficients of these functions are the following

$$a_i = \frac{-3h_i^2}{(h_{i-1}+h_i)(3h_{i-1}+4h_i+3h_{i+1})},\ \bar{a}_i = \frac{-3h_i^2}{(3h_{i-1}+4h_i+3h_{i+1})(h_i+h_{i+1})}$$

$$c_j = \frac{-3k_j^2}{(k_{j-1}+k_j)(3k_{j-1}+4k_j+3k_{j+1})},\ \bar{c}_j = \frac{-3k_j^2}{(3k_{j-1}+4k_j+3k_{j+1})(k_j+k_{j+1})},$$

and $b_{ij} = 1-(a_i+\bar{a}_i+c_j+\bar{c}_j)$, for the operator T_2. It is easy to verify that for all criss-cross triangulations, one has $|a_i|, |\bar{a}_i|, |c_j|, |\bar{c}_j| \leq \frac{3}{4}$. For the operator G_2, we get

$$\alpha_i = \frac{-h_i^2}{(h_{i-1}+h_i)(h_{i-1}+h_i+h_{i+1})},\ \bar{\alpha}_i = \frac{-h_i^2}{(h_{i-1}+h_i+h_{i+1})(h_i+h_{i+1})}$$

$$\gamma_j = \frac{-k_j^2}{(k_{j-1}+k_j)(k_{j-1}+k_j+k_{j+1})},\ \bar{\gamma}_j = \frac{-k_j^2}{(k_{j-1}+k_j+k_{j+1})(k_j+k_{j+1})},$$

and $\beta_{ij} = 1-(\alpha_i+\bar{\alpha}_i+\gamma_j+\bar{\gamma}_j)$. It is easy to verify that for all criss-cross triangulations, one has $|\alpha_i|, |\bar{\alpha}_i|, |\gamma_j|, |\bar{\gamma}_j| \leq 1$. ¿From these inequalities and the fact that

$$\|T_2\|_\infty \leq 1+2\max\{|a_i|, |\bar{a}_i|, |c_j|, |\bar{c}_j|\},\quad \|G_2\|_\infty \leq 1+2\max\{|\alpha_i|, |\bar{\alpha}_i|, |\gamma_j|, |\bar{\gamma}_j|\},$$

one can deduce the following interesting result.

Theorem 6. *For all non-uniform criss-cross triangulations, the operators T_2 and G_2 are uniformly bounded. Moreover,*

$$\|T_2\|_\infty \leq 7 \ \text{and}\ \|G_2\|_\infty \leq 9.$$

Remark. In the case of *uniform* criss-cross triangulations, one gets respectively:

$$a_i = \bar{a}_i = c_j = \bar{c}_j = \frac{-3}{20},\ b_{ij} = \frac{8}{5},\ \Longrightarrow\ \|T_2\|_\infty \leq \frac{11}{5} \approx 2.22$$

$$\alpha_i = \bar{\alpha}_i = \gamma_j = \bar{\gamma}_j = \frac{-1}{6},\ \beta_{ij} = \frac{5}{3},\ \Longrightarrow\ \|G_2\|_\infty \leq \frac{7}{3} \approx 2.33.$$

5.3. **Powell-Sabin quasi-interpolants.** Recently ([25],[26]), using an interesting result by Dierckx [19], we have introduced and studied new families of quadratic splines quasi-interpolants defined on Powell-Sabin type triangulations. As for the previous QIs of this section, we have obtained some families of QIs which are both *exact on* Π_2 and *uniformly bounded* independently of the partition.

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6. Some applications

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6.1. **Approximation of functions.** ¿From a classical result in approximation theory (see e.g. [20]) we know that if Q is an operator defined on a space of smooth functions f with values in a space of splines $\mathcal{S}$, one has

$$\|f - Qf\|_\infty \leq (1 + \|Q\|_\infty) d_\infty(f, \mathcal{S})$$

As the various QIs studied above have uniformly bounded norms, their approximation order is only governed by $d_\infty(f, \mathcal{S})$, i.e. by the distance of f to the maximal space of polynomials included in $\mathcal{S}$. Since the values of $\|Q\|_\infty$ are small, we obtain quite good approximants which can be used in various fields of numerical anlysis.

6.2. **Approximation of zeros.** Quadratic dQIs are simple and good approximants and their zeros are rather easy to compute, so they give good approximations of the zeros of the approximated function. We already did some computations with orthogonal polynomials, and the first results are encouraging (see e.g.[36], [38]).

6.3. **Quadrature formulas.** For the same reasons, quadrature formulas (QF) are easily obtained by integrating spline QIs. An interesting univariate example is given in [36]. The study of bivariate and trivariate QF is still in progress and the results already obtained in two and three variables are also encouraging [18].

6.4. **Pseudo-spectral methods associated with quasi-interpolants.** Derivatives of QIs give quite good approximations of derivatives of the approximated function. This simple fact is the basic idea for developing pseudo-spectral methods based on univariat and multivariate dQIs with low degrees.

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